CHAPTER 4 PROBLEMS AND EXERCISES SOLUTIONS

Table 1 Generic aluminum alloy (E = 70 GPa, ν = 0.33) stiffness matrix and density

$$\mathbf{C}' = \begin{bmatrix} 103.7 & 51.1 & 51.1 & 0 & 0 & 0 \\ 51.1 & 103.7 & 51.1 & 0 & 0 & 0 \\ 51.1 & 51.1 & 103.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 26.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 26.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 26.3 \end{bmatrix} \text{GPa}, \qquad \rho = 2700 \text{ kg/m}^3$$

Table 2 7075-T6 aluminum alloy (E = 71.7 GPa, ν = 0.33) stiffness matrix and density

$$\mathbf{C}' = \begin{bmatrix} 106.2 & 52.3 & 52.3 & 0 & 0 & 0 \\ 52.3 & 106.2 & 52.3 & 0 & 0 & 0 \\ 52.3 & 52.3 & 106.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 26.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 26.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 26.7 \end{bmatrix} \text{GPa}, \qquad \rho = 2810 \text{ kg/m}^3$$

Table 3 T300-914 CFRP unidirectional composite stiffness matrix and density

$$\mathbf{C}' = \begin{bmatrix} 143.8 & 6.2 & 6.2 & 0 & 0 & 0 \\ 6.2 & 13.3 & 6.5 & 0 & 0 & 0 \\ 6.2 & 6.5 & 13.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.7 \end{bmatrix} \text{GPa}, \qquad \rho = 1560 \text{ kg/m}^3$$

Table 4 Fully orthotropic CFRP composite stiffness matrix and density

$$\mathbf{C}' = \begin{bmatrix} 70 & 23.9 & 6.2 & 0 & 0 & 0 \\ 23.9 & 33 & 6.8 & 0 & 0 & 0 \\ 6.2 & 6.8 & 14.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 21.9 \end{bmatrix} \text{GPa}, \qquad \rho = 1500 \text{ kg/m}^3$$

PROBLEM 1: CHRISTOFFEL EQUATION FOR BULK WAVES

Consider a plane wave propagating through an isotropic material of stiffness matrix $\mathbf{C}$ in a direction contained in the vertical plane $x_1 O x_3$ and rotated by an angle θ about the x_2 axis. Do the following:

(a) Calculate the stiffness tensor $\mathbf{c}$
(b) Calculate the acoustic tensor $\mathbf{\Gamma}$
(c) State the Christoffel equation and find its eigenvalues
(d) Find the wavespeeds corresponding to these eigenvalues
(e) Find the polarization direction for each wavespeed
(f) Identify any pure waves that might show up during the analysis

Solution:

(a) Calculate the stiffness tensor $\mathbf{c}$: The stiffness tensor $\mathbf{c}$ is obtained from the stiffness matrix $\mathbf{C}$ using the correspondence formulae given in Eq. (2.35). This conversion was discussed and exemplified in Problem 2.2 of Chapter 2. In this problem, the stiffness matrix $\mathbf{C}$ is taken to be the material stiffness matrix $\mathbf{C}'$, i.e., $\mathbf{C} = \mathbf{C}'$.

(b) Calculate the acoustic tensor $\mathbf{\Gamma}$. The acoustic tensor $\mathbf{\Gamma}$ is a 3×3 matrix given by Eq. (4.16), i.e.,

$$\mathbf{\Gamma} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{12} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} \end{bmatrix} \tag{4.16}$$

The elements Γ_{im} of $\mathbf{\Gamma}$ are defined by Eq. (4.12) in terms of the elements of the stiffness tensor $\mathbf{c} = c_{ijkm}$ and the direction cosines n_j of the wavefront direction, $\vec{n} = n_1\vec{e}_1 + n_2\vec{e}_2 + n_3\vec{e}_3$, i.e.,

$$\Gamma_{im} = \sum_{j=1}^{3}\sum_{k=1}^{3} c_{ijkm} n_k n_j \qquad i,j,k,m = 1,2,3 \tag{4.12}$$

For this problem, the direction cosines are $n_1 = \cos\theta$, $n_2 = 0$, $n_3 = \sin\theta$.

(c) State the Christoffel equation: The Christoffel equation for bulk composite is given by Eq. (4.19), i.e.,

$$\mathbf{\Gamma}\hat{\mathbf{u}} = \lambda\hat{\mathbf{u}} \tag{4.19}$$

Find its eigenvalues: Eq. (4.19) represents an algebraic eigenvalue problem for which numerical tools are readily available. Solution of Eq. (4.19) yields three eigenvalues λ_I, λ_{II}, λ_{III}, and the corresponding eigenvectors $\hat{\mathbf{u}}_I$, $\hat{\mathbf{u}}_{II}$, $\hat{\mathbf{u}}_{III}$.

(d) Find the wavespeeds corresponding to these eigenvalues: Recall the relation $\lambda = \rho v^2$. Hence, the wavespeed is calculated as $v = \sqrt{\lambda/\rho}$

(e) Find the polarization direction for each wavespeed: the polarization direction for a given wavespeed v is defined by the eigenvector $\hat{\mathbf{u}}$ corresponding to the eigenvalue λ that generated that wavespeed.

(f) Identify pure waves: pure waves are the waves that have the polarization direction aligned with the wave propagation direction.

PROBLEM 2: CHRISTOFFEL EQUATION IN BULK ISOTROPIC MATERIAL

The Christoffel equation was developed for anisotropic materials. However, in this problem, we will apply the Christoffel equation approach to an isotropic material in order to establish that the formalism developed for wave propagation in anisotropic materials can be also used for wave propagation in isotropic materials and gives the expected results.

Consider a plane wave propagating through an isotropic material of elastic constants E and ν. The wave propagates in a direction contained in the vertical plane $x_1 O x_3$ and rotated by an angle θ about the x_2 axis. Do the following:

(a) Calculate the stiffness matrix $\mathbf{C}$ and the stiffness tensor $\mathbf{c}$
(b) Calculate the acoustic tensor $\mathbf{\Gamma}$
(c) Find the eigenvalues of the Christoffel equation
(d) Find the wavespeeds corresponding to these eigenvalues
(e) Find the polarization direction for each wavespeed
(f) Identify any pure waves that might show up during the analysis
(g) Discuss your results

<u>Numerical values</u>:
Wavefront direction angle $\theta = 0°, 10°, 15°, 30°, 45°, 60°, 75°, 90°$.
Material: 7075-T6 aluminum, Table 2

<u>Solution</u>:

(a) to find the stiffness matrix $\mathbf{C}$, recall Eq. (2.47) of Chapter 2, i.e.,

$$\mathbf{C}^{isotropic} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{2.47}$$

The Lame constants λ, μ are calculated with Eq. (2.45), i.e.,

$$\lambda = \frac{\nu}{(1+\nu)(1-2\nu)} E \qquad \mu = G = \frac{1}{2(1+\nu)} E \qquad \text{(Lame constants)} \tag{2.45}$$

Upon calculation, one gets,

```
7075-T6 aluminum results
stiffness matrix, GPa =
  106.2340   52.3242   52.3242         0         0         0
   52.3242  106.2340   52.3242         0         0         0
   52.3242   52.3242  106.2340         0         0         0
         0         0         0   26.9549         0         0
         0         0         0         0   26.9549         0
         0         0         0         0         0   26.9549
```

The stiffness tensor $\mathbf{c}$ is calculated as shown in Problem 1.
The theory for items (b) through (f) is same as presented in Problem 1
Upon calculation (see MATLAB code), one gets the following numerical results:

$\theta = 0°$

```
theta = 0 deg
wavefront direction =
     1     0     0
(b) acoustic tensor, Gamma, GPa =
  106.2340         0         0
         0   26.9549         0
         0         0   26.9549
(c) eigenvalues, GPa =
  106.2340   26.9549   26.9549
(d) wavespeeds, v, km/s =
    6.1486    3.0972    3.0972
(e) polarization vectors =
     1     0     0
     0     0     1
     0     1     0
```

$\theta = 10°$

```
theta = 10 deg
wavefront direction =
    0.9848         0    0.1736
(b) acoustic tensor, Gamma, GPa =
  103.8434         0   13.5575
         0   26.9549         0
   13.5575         0   29.3454
(c) eigenvalues, GPa =
  106.2340   26.9549   26.9549
(d) wavespeeds, v, km/s =
    6.1486    3.0972    3.0972
(e) polarization vectors =
    0.9848   -0.1736         0
         0         0    1.0000
    0.1736    0.9848         0
```

$\theta = 15°$

```
theta = 15 deg
wavefront direction =
    0.9659         0    0.2588
(b) acoustic tensor, Gamma, GPa =
  100.9233         0   19.8198
         0   26.9549         0
   19.8198         0   32.2656
(c) eigenvalues, GPa =
  106.2340   26.9549   26.9549
(d) wavespeeds, v, km/s =
    6.1486    3.0972    3.0972
(e) polarization vectors =
    0.9659         0   -0.2588
         0    1.0000         0
    0.2588         0    0.9659
```

$\theta = 30°$

```
theta = 30 deg
wavefront direction =
    0.8660         0    0.5000
(b) acoustic tensor, Gamma, GPa =
   86.4142         0   34.3288
         0   26.9549         0
   34.3288         0   46.7747
(c) eigenvalues, GPa =
  106.2340   26.9549   26.9549
(d) wavespeeds, v, km/s =
    6.1486    3.0972    3.0972
(e) polarization vectors =
    0.8660   -0.5000         0
         0         0    1.0000
    0.5000    0.8660         0
```

$\theta = 45°$

```
theta = 45 deg
wavefront direction =
    0.7071         0    0.7071
(b) acoustic tensor, Gamma, GPa =
   66.5944         0   39.6395
         0   26.9549         0
   39.6395         0   66.5944
(c) eigenvalues, GPa =
  106.2340   26.9549   26.9549
(d) wavespeeds, v, km/s =
    6.1486    3.0972    3.0972
(e) polarization vectors =
    0.7071   -0.7071         0
         0         0    1.0000
    0.7071    0.7071         0
```

$\theta = 60°$

```
theta = 60 deg
wavefront direction =
    0.5000         0    0.8660
(b) acoustic tensor, Gamma, GPa =
   46.7747         0   34.3288
         0   26.9549         0
   34.3288         0   86.4142
(c) eigenvalues, GPa =
  106.2340   26.9549   26.9549
(d) wavespeeds, v, km/s =
    6.1486    3.0972    3.0972
(e) polarization vectors =
    0.5000    0.8660         0
         0         0    1.0000
    0.8660   -0.5000         0
```

$\theta = 75°$

```
theta = 75 deg
wavefront direction =
    0.2588         0    0.9659
(b) acoustic tensor, Gamma, GPa =
   32.2656         0   19.8198
         0   26.9549         0
   19.8198         0  100.9233
(c) eigenvalues, GPa =
  106.2340   26.9549   26.9549
(d) wavespeeds, v, km/s =
    6.1486    3.0972    3.0972
(e) polarization vectors =
    0.2588         0    0.9659
         0    1.0000         0
    0.9659         0   -0.2588
```

$\theta = 90°$

```
theta = 90 deg
wavefront direction =
    0.0000         0    1.0000
(b) acoustic tensor, Gamma, GPa =
   26.9549         0    0.0000
         0   26.9549         0
    0.0000         0  106.2340
(c) eigenvalues, GPa =
  106.2340   26.9549   26.9549
(d) wavespeeds, v, km/s =
    6.1486    3.0972    3.0972
(e) polarization vectors =
     0     0     1
     0     1     0
     1     0     0
```

=====================

(f) Identify any pure waves that might show up in the analysis:

The numerical results shown that, for all wave propagation directions $\mathbf{n}$, the first polarization direction is parallel to the propagation direction, i.e., $\hat{\mathbf{u}}_I = \mathbf{n}$. This means that the first polarization direction is a pure wave because it is aligned with the wave propagation direction. This is the P wave of this isotropic material; the corresponding wavespeed is $v_I = v_P = 6.1486 \text{ km/s}$.

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(g) Discuss your results:

(1) In an isotropic material, we expect only two wavespeeds, the pressure (P) wave and the shear (S) wave. We expect the polarization corresponding to the P wave to be along the propagation direction. Regarding the S wave, we expect two mutually orthogonal polarizations that are also orthogonal to the P wave polarization. These are the shear horizontal (SH) and shear vertical (SV) waves. The numerical results calculated with this Christoffel equation algorithm confirm our expectations.

(2) One of the three polarization vectors is always in the x_2 direction, i.e., $[0\ 1\ 0]$ for all angles. This is due to the fact that the wave propagation direction is contained in the vertical plane $x_1 O x_3$ and hence the wavefront normal $\vec{n} = n_1\vec{e}_1 + n_2\vec{e}_2 + n_3\vec{e}_3$ does not have components in the x_2 direction. Hence the u_2 motion is always decoupled from the u_1-u_3 motion. This $[0\ 1\ 0]$ polarization vector corresponds to SH wave which is decoupled from the other two waves. (The $[0\ 1\ 0]$ polarization appear sometimes as the second polarization vector and sometimes as the third; this is to be expected since the second and third wavespeeds are equal and hence their polarization vectors are interchangeable.)

(2) The other two polarization vectors correspond to the P wave and the SV wave, respectively.

(3) For all angles, the P wave is a pure wave because it is always parallel to the propagation direction.

(4) For wave angles 0° and 90°, the P and SV waves are decoupled, $[1\ 0\ 0]$ and $[0\ 0\ 1]$ respectively for 0°; $[0\ 0\ 1]$ and $[1\ 0\ 0]$ respectively for $90°$. For angles different from 0° and 90°, the P and SV waves are coupled, e.g., $[0.866\quad 0\quad 0.500]$ and $[-0.500\quad 0\quad 0.866]$ respectively for 30°.

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PROBLEM 3: CHRISTOFFEL EQUATION IN BULK UNIDIRECTIONAL COMPOSITE

Consider a plane wave propagating through an unidirectional composite material of stiffness matrix $\mathbf{C}$. The wave propagates in the vertical plane $x_1 O x_3$ and rotated by an angle θ about the x_2 axis. Do the following:

(a) Calculate the stiffness tensor $\mathbf{c}$
(b) Calculate the acoustic tensor $\mathbf{\Gamma}$
(c) Find the eigenvalues of the Christoffel equation
(d) Find the wavespeeds corresponding to these eigenvalues
(e) Find the polarization direction for each wavespeed
(f) Identify any pure waves that might show up during the analysis
(g) Discuss your results

Numerical values:
Wavefront direction angle $\theta = 0°, 10°, 15°, 30°, 45°, 60°, 75°, 90°$.
T300/914 CFRP, Table 3

Solution:

```
unidirectional T300-914 CFRP results
(a) stiffness matrix, GPa =
  143.8000    6.2000    6.2000         0         0         0
    6.2000   13.3000    6.5000         0         0         0
    6.2000    6.5000   13.3000         0         0         0
         0         0         0    3.4000         0         0
         0         0         0         0    5.7000         0
         0         0         0         0         0    5.7000
```

The stiffness tensor $\mathbf{c}$ is calculated as shown in Problem 1.
The theory for items (b) through (f) is same as presented in Problem 1
Upon calculation (see MATLAB code), one gets the following numerical results:

$\theta = 0°$

```
theta = 0 deg
wavefront direction =
     1     0     0
(b) acoustic tensor, Gamma, GPa =
  143.8000         0         0
         0    5.7000         0
         0         0    5.7000
(c) eigenvalues, GPa =
  143.8000    5.7000    5.7000
(d) wavespeeds, v, km/s =
    9.6010    1.9115    1.9115
(e) polarization vectors =
     1     0     0
     0     0     1
     0     1     0
```

$\theta = 10°$

```
theta = 10 deg
wavefront direction =
    0.9848         0    0.1736
(b) acoustic tensor, Gamma, GPa =
  139.6358         0    2.0350
         0    5.6306         0
    2.0350         0    5.9292
(c) eigenvalues, GPa =
  139.6667    5.8982    5.6306
(d) wavespeeds, v, km/s =
    9.4620    1.9445    1.8998
(e) polarization vectors =
    0.9999   -0.0152         0
         0         0    1.0000
    0.0152    0.9999         0
```

$\theta = 15°$

```
theta = 15 deg
wavefront direction =
    0.9659         0    0.2588
(b) acoustic tensor, Gamma, GPa =
  134.5491         0    2.9750
         0    5.5459         0
    2.9750         0    6.2091
(c) eigenvalues, GPa =
  134.6180    6.1402    5.5459
(d) wavespeeds, v, km/s =
    9.2894    1.9839    1.8855
(e) polarization vectors =
    0.9997   -0.0232         0
         0         0    1.0000
    0.0232    0.9997         0
```

$\theta = 30°$

```
theta = 30 deg
wavefront direction =
    0.8660         0    0.5000
(b) acoustic tensor, Gamma, GPa =
  109.2750         0    5.1529
         0    5.1250         0
    5.1529         0    7.6000
(c) eigenvalues, GPa =
  109.5355    7.3395    5.1250
(d) wavespeeds, v, km/s =
    8.3794    2.1691    1.8125
(e) polarization vectors =
    0.9987   -0.0505         0
         0         0    1.0000
    0.0505    0.9987         0
```

$\theta = 45^\circ$

```
theta = 45 deg
wavefront direction =
    0.7071         0    0.7071
(b) acoustic tensor, Gamma, GPa =
   74.7500         0    5.9500
         0    4.5500         0
    5.9500         0    9.5000
(c) eigenvalues, GPa =
   75.2881    8.9619    4.5500
(d) wavespeeds, v, km/s =
    6.9471    2.3968    1.7078
(e) polarization vectors =
    0.9959   -0.0901         0
         0         0    1.0000
    0.0901    0.9959         0
```

$\theta = 60^\circ$

```
theta = 60 deg
wavefront direction =
    0.5000         0    0.8660
(b) acoustic tensor, Gamma, GPa =
   40.2250         0    5.1529
         0    3.9750         0
    5.1529         0   11.4000
(c) eigenvalues, GPa =
   41.1184   10.5066    3.9750
(d) wavespeeds, v, km/s =
    5.1340    2.5952    1.5963
(e) polarization vectors =
    0.9853   -0.1708         0
         0         0    1.0000
    0.1708    0.9853         0
```

$\theta = 75^\circ$

```
theta = 75 deg
wavefront direction =
    0.2588         0    0.9659
(b) acoustic tensor, Gamma, GPa =
   14.9509         0    2.9750
         0    3.5541         0
    2.9750         0   12.7909
(c) eigenvalues, GPa =
   17.0359   10.7059    3.5541
(d) wavespeeds, v, km/s =
    3.3046    2.6197    1.5094
(e) polarization vectors =
    0.8189   -0.5739         0
         0         0    1.0000
    0.5739    0.8189         0
```

$\theta = 90^\circ$

```
theta = 90 deg
wavefront direction =
    0.0000         0    1.0000
(b) acoustic tensor, Gamma, GPa =
    5.7000         0    0.0000
         0    3.4000         0
    0.0000         0   13.3000
(c) eigenvalues, GPa =
   13.3000    5.7000    3.4000
(d) wavespeeds, v, km/s =
    2.9199    1.9115    1.4763
(e) polarization vectors =
     0     1     0
     0     0     1
     1     0     0
```

(f) Identify any pure waves that might show up in the analysis:
Two pure waves have been identified as follows:

(1) For $\theta = 0°$, the wave propagation direction is $\vec{n} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. This wavefront orientation contains a pure wave because one of the polarization directions, specifically $\hat{\mathbf{u}}_I = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ is aligned with the wave propagation direction $\vec{n} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$; the corresponding wavespeed is $v_I = 9.6010 \text{ km/s}$.

(2) For $\theta = 90°$, the wave propagation direction is $\vec{n} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. This wavefront orientation contains a pure wave because one of the polarization directions, specifically $\hat{\mathbf{u}}_I = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ is aligned with the wave propagation direction $\vec{n} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$; the corresponding wavespeed is $v_I = 2.9199 \text{ km/s}$.

=============

(g) Discuss your results:

(1) Because the wave propagation direction is contained in the vertical plane $x_1 O x_3$, the wavefront normal $\vec{n} = n_1\vec{e}_1 + n_2\vec{e}_2 + n_3\vec{e}_3$ does not have components in the x_2 direction. Hence the u_2 motion is always decoupled from the u_1-u_3 motion. The third polarization vector is always in the x_2 direction, i.e., $\hat{\mathbf{u}}_{III} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$; this is true for all angles. This third polarization vector corresponds to a quasi SH wave decoupled from the other two waves

(2) The first and second polarization vectors correspond to a quasi P wave and a quasi SV wave, respectively. This is especially clear at low θ values where the quasi P wavespeed is much larger than the quasi SV wavespeed (e.g., ~8.38 km/s vs. ~2.17 km/s at $\theta = 30°$.)

(3) For wave angles 0° and 90°, the quasi P wave is a pure wave because it is parallel to the propagation direction; for the other angles, the quasi P wave deviates from the wave propagation direction, i.e., it is not a pure wave.

(4) For wave angles 0° and 90°, the P and SV waves are decoupled. For angles different from 0° and 90°, the quasi P and quasi SV waves are coupled.

(5) At 0° angle, the quasi SH and quasi SV waves have the same wavespeeds $v_{II} = v_{III} = 1.9115 \text{ km/s}$ because the material is symmetric about the x_1 axis.

(6) For all the nonzero angles, $\theta \neq 0°$, the quasi SH wave has always the lowest speed

PROBLEM 4: CHRISTOFFEL EQUATION IN BULK ORTHOTROPIC COMPOSITE

Consider a plane wave propagating through an unidirectional composite material of stiffness matrix $\mathbf{C}$. The wave propagates in the vertical plane $x_1 O x_3$ and rotated by an angle θ about the x_2 axis. Do the following:

(a) Calculate the stiffness tensor $\mathbf{c}$
(b) Calculate the acoustic tensor $\boldsymbol{\Gamma}$
(c) Find the eigenvalues of the Christoffel equation
(d) Find the wavespeeds corresponding to these eigenvalues
(e) Find the polarization direction for each wavespeed
(f) Identify any pure waves that might show up during the analysis
(g) Discuss your results

Numerical values:
Wavefront direction angle $\theta = 0°, 10°, 15°, 30°, 45°, 60°, 75°, 90°$.
Fully orthotropic CFRP, Table 4

Solution

```
Fuly orthotropic CFRP results
(a) stiffness matrix, GPa =
   70.0000   23.9000    6.2000         0         0         0
   23.9000   33.0000    6.8000         0         0         0
    6.2000    6.8000   14.7000         0         0         0
         0         0         0    4.2000         0         0
         0         0         0         0    4.7000         0
         0         0         0         0         0   21.9000
```

The stiffness tensor $\mathbf{c}$ is calculated as shown in Problem 1.
The theory for items (b) through (f) is same as presented in Problem 1
Upon calculation (see MATLAB code), one gets the following numerical results:

$\theta = 0^\circ$

```
theta = 0 deg
wavefront direction =
     1     0     0
(b) acoustic tensor, Gamma, GPa =
   70.0000         0         0
         0   21.9000         0
         0         0    4.7000
(c) eigenvalues, GPa =
   70.0000   21.9000    4.7000
(d) wavespeeds, v, km/s =
    6.8313    3.8210    1.7701
(e) polarization vectors =
     1     0     0
     0     1     0
     0     0     1
```

$\theta = 10^\circ$

```
theta = 10 deg
wavefront direction =
    0.9848         0    0.1736
(b) acoustic tensor, Gamma, GPa =
   68.0310         0    1.8640
         0   21.3663         0
    1.8640         0    5.0015
(c) eigenvalues, GPa =
   68.0860   21.3663    4.9465
(d) wavespeeds, v, km/s =
    6.7373    3.7741    1.8159
(e) polarization vectors =
    0.9996         0   -0.0295
         0    1.0000         0
    0.0295         0    0.9996
```

$\theta = 15^\circ$

```
theta = 15 deg
wavefront direction =
    0.9659         0    0.2588
(b) acoustic tensor, Gamma, GPa =
   65.6257         0    2.7250
         0   20.7143         0
    2.7250         0    5.3699
(c) eigenvalues, GPa =
   65.7487   20.7143    5.2469
(d) wavespeeds, v, km/s =
    6.6206    3.7161    1.8703
(e) polarization vectors =
    0.9990         0   -0.0451
         0    1.0000         0
    0.0451    0.0000    0.9990
```

$\theta = 30^\circ$

```
theta = 30 deg
wavefront direction =
    0.8660         0    0.5000
(b) acoustic tensor, Gamma, GPa =
   53.6750         0    4.7198
         0   17.4750         0
    4.7198         0    7.2000
(c) eigenvalues, GPa =
   54.1495   17.4750    6.7255
(d) wavespeeds, v, km/s =
    6.0083    3.4132    2.1175
(e) polarization vectors =
    0.9950         0   -0.1000
         0    1.0000         0
    0.1000         0    0.9950
```

$\theta = 45°$	$\theta = 60°$
theta = 45 deg wavefront direction = 0.7071 0 0.7071 (b) acoustic tensor, Gamma, GPa = 37.3500 0 5.4500 0 13.0500 0 5.4500 0 9.7000 (c) eigenvalues, GPa = 38.3855 13.0500 8.6645 (d) wavespeeds, v, km/s = 5.0587 2.9496 2.4034 (e) polarization vectors = 0.9824 0 -0.1867 0 1.0000 0 0.1867 0 0.9824	theta = 60 deg wavefront direction = 0.5000 0 0.8660 (b) acoustic tensor, Gamma, GPa = 21.0250 0 4.7198 0 8.6250 0 4.7198 0 12.2000 (c) eigenvalues, GPa = 23.0737 10.1513 8.6250 (d) wavespeeds, v, km/s = 3.9220 2.6014 2.3979 (e) polarization vectors = 0.9173 -0.3982 0 0 0 1.0000 0.3982 0.9173 0
$\theta = 75°$	$\theta = 90°$
theta = 75 deg wavefront direction = 0.2588 0 0.9659 (b) acoustic tensor, Gamma, GPa = 9.0743 0 2.7250 0 5.3857 0 2.7250 0 14.0301 (c) eigenvalues, GPa = 15.2354 7.8690 5.3857 (d) wavespeeds, v, km/s = 3.1870 2.2904 1.8948 (e) polarization vectors = 0.4045 0.9145 0 0 0 1.0000 0.9145 -0.4045 0.0000	theta = 90 deg wavefront direction = 0.0000 0 1.0000 (b) acoustic tensor, Gamma, GPa = 4.7000 0 0.0000 0 4.2000 0 0.0000 0 14.7000 (c) eigenvalues, GPa = 14.7000 4.7000 4.2000 (d) wavespeeds, v, km/s = 3.1305 1.7701 1.6733 (e) polarization vectors = 0 1 0 0 0 1 1 0 0

(f) Identify any pure waves that might show up in the analysis:

Two pure waves have been identified as follows:

(1) For $\theta = 0°$, the wave propagation direction is $\vec{n} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$. This wavefront orientation contains a pure wave because one of the polarization directions, specifically $\hat{\mathbf{u}}_I = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$ is aligned with the wave propagation direction $\vec{n} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$; the corresponding wavespeed is $v_I = 6.8313$ km/s.

(2) For $\theta = 90°$, the wave propagation direction is $\vec{n} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$. This wavefront orientation contains a pure wave because one of the polarization directions, specifically $\hat{\mathbf{u}}_I = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$ is aligned with the wave propagation direction $\vec{n} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$; the corresponding wavespeed is $v_I = 3.1305$ km/s.

=============

(g) Discuss your results:

(1) Because the wave propagation direction is contained in the vertical plane $x_1 O x_3$, the wavefront normal $\vec{n} = n_1\vec{e}_1 + n_2\vec{e}_2 + n_3\vec{e}_3$ does not have components in the x_2 direction. Hence the u_2 motion is always decoupled from the u_1-u_3 motion and one polarization vector is always in the x_2 direction; this is true for all angles. This polarization vector corresponds to a quasi SH wave $\hat{\mathbf{u}}_{SH} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}$ decoupled from the other two waves

(2) The other two polarization vectors correspond to a quasi P wave and a quasi SV wave, respectively.

(3) For wave angles 0° and 90°, the quasi P wave is a pure wave because it is parallel to the propagation direction; for the other angles, the quasi P wave deviates from the wave propagation direction, i.e., it is not a pure wave.

(4) For wave angles 0° and 90°, the P and SV waves are decoupled. For angles different from 0° and 90°, the quasi P and quasi SV waves are coupled.

(5) The quasi SV wave has the lowest speed for small angles; at higher angles, the quasi SH wave has the lowest speed. The switch over between SV and SH takes place between 45° and 60°. An iterative search revealed that the actual switch over takes place at $\theta = 55.567°$ where $v_{SH} = v_{SV} = 2.5637$ km/s

PROBLEM 5: LAMINA CHRISTOFFEL EQUATION FOR ISOTROPIC LAMINA

Given an isotropic lamina of thickness *h*, solve the lamina Christoffel equation for guided wave propagation along the fiber direction x_1. Do the following:

(a) For a given wavespeed v, set up the cubic equation α^2, where α^2 are the partial-wave slowness coefficients

(b) Solve the cubic equation, find the three α^2 roots $\alpha_I^2, \alpha_{II}^2, \alpha_{III}^2$, and the corresponding six eigenvalues $\alpha^{(1)} = +\alpha_I$, $\alpha^{(2)} = -\alpha_I$, $\alpha^{(3)} = +\alpha_{II}$, $\alpha^{(4)} = -\alpha_{II}$, $\alpha^{(5)} = +\alpha_{III}$, $\alpha^{(6)} = -\alpha_{III}$

(c) Substitute each eigenvalue $\alpha^{(i)}, i = 1,...,6$ into the Christoffel matrix and find the corresponding eigenvectors $\mathbf{U}^{(i)},\ i = 1,...,6$

(d) Repeat for other values of v

(e) Discuss your findings

Numerical example: 7075-T6 aluminum alloy, Table 2

<u>Solution</u>

(a) Setup the cubic equation in partial-wave slowness coefficients α^2: Recall Eq. (4.37), i.e.,

$$A_6\alpha^6 + A_4\alpha^4 + A_2\alpha^2 + A_0 = 0 \tag{4.37}$$

with the coefficients A_6, A_4, A_2, A_0 given by Eqs. (4.38)—(4.41), i.e.,

$$A_6 = C_{33}C_{44}C_{55} - C_{33}C_{45}^2 \tag{4.38}$$

$$\begin{aligned} A_4 &= (C_{55}C_{44} - C_{45}^2)(C_{55} - \rho v^2) + C_{55}C_{33}(C_{66} - \rho v^2) + C_{44}C_{33}(C_{11} - \rho v^2) \\ &\quad -2C_{16}C_{45}C_{33} + 2\left(C_{45} + C_{36}\right)\left(C_{13} + C_{55}\right)C_{45} - \left(C_{13} + C_{55}\right)^2 C_{44} - \left(C_{45} + C_{36}\right)^2 C_{55} \end{aligned} \tag{4.39}$$

$$\begin{aligned} A_2 &= C_{33}(C_{11} - \rho v^2)(C_{66} - \rho v^2) + C_{44}(C_{11} - \rho v^2)(C_{55} - \rho v^2) + C_{55}(C_{66} - \rho v^2)(C_{55} - \rho v^2) \\ &\quad -(C_{11} - \rho v^2)\left(C_{45} + C_{36}\right)^2 - (C_{66} - \rho v^2)\left(C_{13} + C_{55}\right)^2 - 2(C_{55} - \rho v^2)C_{16}C_{45} \\ &\quad +2C_{16}\left(C_{45} + C_{36}\right)\left(C_{13} + C_{55}\right) - C_{16}^2 C_{33} \end{aligned} \tag{4.40}$$

$$A_0 = \left[(C_{11} - \rho v^2)(C_{66} - \rho v^2) - C_{16}^2\right](C_{55} - \rho v^2) \tag{4.41}$$

(b) Solve Eq. (4.37) using a polynomial roots algorithm available in MATLAB to get $\alpha_I^2, \alpha_{II}^2, \alpha_{III}^2$; then, separate the six individual eigenvalues $\alpha^{(1)} = +\alpha_I$, $\alpha^{(2)} = -\alpha_I$, $\alpha^{(3)} = +\alpha_{II}$, $\alpha^{(4)} = -\alpha_{II}$, $\alpha^{(5)} = +\alpha_{III}$, $\alpha^{(6)} = -\alpha_{III}$.

Because the material is isotropic, the solution has a pair of double roots $\alpha_I = \alpha_{II} = \alpha_S$ since, for isotropic materials, there is no difference between the SH and SV shear waves that propagate with the same wavespeed which is the shear wavespeed in the isotropic material. The third root correspond to the pressure wave, i.e., $\alpha_{III} = \alpha_P$. Then, the six individual α's are $\alpha^{(1)} = +\alpha_S$, $\alpha^{(2)} = -\alpha_S$, $\alpha^{(3)} = +\alpha_S$, $\alpha^{(4)} = -\alpha_S$, $\alpha^{(5)} = +\alpha_P$, $\alpha^{(6)} = -\alpha_P$

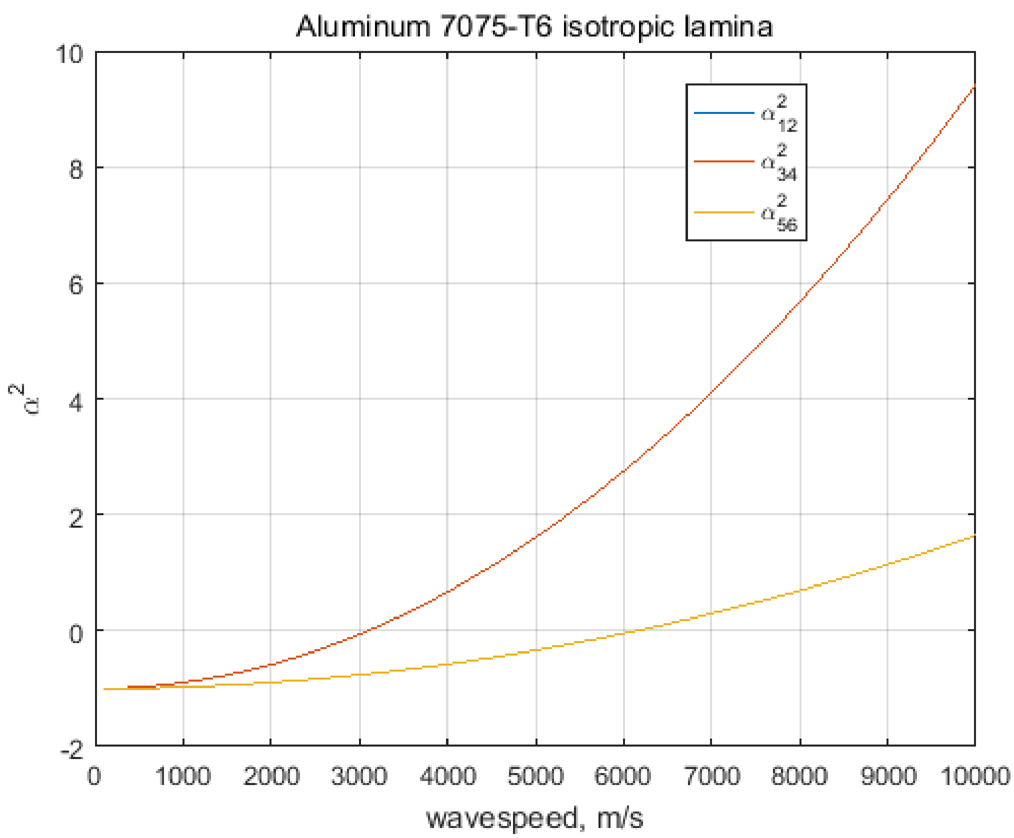
(c) Substitute each eigenvalue $\alpha^{(i)}$, $i = 1,\ldots,6$ into the Christoffel matrix of Eq. (4.35), i.e.,

$$\begin{bmatrix} \left(C_{11} - \rho v^2\right) + C_{55}\alpha^2 & C_{16} + C_{45}\alpha^2 & \left(C_{13} + C_{55}\right)\alpha \\ C_{16} + C_{45}\alpha^2 & \left(C_{66} - \rho v^2\right) + C_{44}\alpha^2 & \left(C_{36} + C_{45}\right)\alpha \\ \left(C_{13} + C_{55}\right)\alpha & \left(C_{36} + C_{45}\right)\alpha & \left(C_{55} - \rho v^2\right) + C_{33}\alpha^2 \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.35}$$

Find the corresponding eigenvector $\mathbf{U}^{(i)}$, $i = 1,\ldots,6$ using a standard algebraic eigen problem algorithm (e.g., eig) or a singular value decomposition (SVD) algorithm and choosing the eigenvector corresponding to the eigenvalue which has a value zero.

Note that $\alpha^{(1)} = \alpha^{(3)} = +\alpha_S$; This means that the Christoffel matrix will have two zero-valued eigenvalues, with the corresponding eigenvectors being orthogonal to each other. These are the SH and SV polarization vectors which are orthogonal to each other. Similarly, for the other SH and SV pair, $\alpha^{(2)} = \alpha^{(4)} = -\alpha_S$.

(d) The above analysis is repeated for all wavespeed v values. The plot below shows the variation of α^2 with wavespeeds.



(e) Discussion:

<u>The α^2 Curves</u>

(e1) Examination of the α^2 curves reveals three monotonically increasing curves. Two of these curves, α_{12}^2 and α_{34}^2 are overlapped because they represent double roots corresponding to SV, SH having same wavespeeds in the isotropic material. The third curve, α_{56}^2, corresponds to the P wave.

(e2) All three α^2 curves seem to start from same values at almost zero wavespeed[1] ($v \approx 0$) and then spread out, the S curves α_{12}^2, α_{34}^2 growing faster than the P curve α_{56}^2.

(e3) The S curves cross the horizontal axis at a wavespeed $v_S \approx 3{,}000$ m/s. The P curve crosses the horizontal axis at a higher wavespeed $v_P \approx 6{,}000$ m/s.

<u>The Polarization Vectors</u>

(e4) Examination of the polarization vectors permits the identification of the SH wave associated with α_{34}^2. The SH wave is polarized in the x_2 direction, i.e., its polarization vector only contains the u_2 component. This happens at all wavespeeds.

```
val(:,:,1) =
   0.0000 - 0.9995i   0.0000 + 0.9995i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 + 0.0000i   0.0000 + 0.9999i   0.0000 - 0.9999i
val(:,:,2) =
   0.0000 - 0.9979i   0.0000 + 0.9979i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.9995i   0.0000 - 0.9995i
val(:,:,3) =
   0.0000 - 0.9916i   0.0000 + 0.9916i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.9979i   0.0000 - 0.9979i
```

<u>SH polarization</u>

(e5) At all wavespeeds, the SV wave α_{12}^2 and the P wave α_{56}^2, have polarization vectors aligned with the vertical plane $x_1 \mathrm{O} x_3$ because they contain only the u_1, u_3 displacements. Note that these vectors are complex at speeds below the crossover speeds $v_S \approx 3{,}000$ m/s, $v_P \approx 6{,}000$ m/s.

<u>complex SV polarization vectors</u> <u>complex P polarization vectors</u>

```
val(:,:,1) =
   0.0000 - 0.9995i   0.0000 + 0.9995i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 + 0.0000i   0.0000 + 0.9999i   0.0000 - 0.9999i
val(:,:,2) =
   0.0000 - 0.9979i   0.0000 + 0.9979i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.9995i   0.0000 - 0.9995i
val(:,:,3) =
   0.0000 - 0.9916i   0.0000 + 0.9916i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.9979i   0.0000 - 0.9979i
```

[1] exact $v = 0$ is not computable correctly; here, we took $v_{\min} = 100$ m/s

(e6) As the wavespeed passes the S crossover point $v_S \approx 3{,}000$ m/s, the SV polarization vectors corresponding to α_1, α_2 switch from complex to real values.

complex SV vectors below the S crossover speed

```
val(:,:,16) =
   0.0000 - 0.2485i   0.0000 + 0.2485i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.8729i   0.0000 - 0.8729i
val(:,:,17) =
  -0.2598 + 0.0000i   0.2598 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.8539i   0.0000 - 0.8539i
```

real valued SV vectors above the S crossover speed

(e7) As the wavespeed passes the P crossover point $v_P \approx 6{,}000$ m/s, the P polarization vectors corresponding to α_5, α_6 switch from complex to real values.

complex P vectors below the P crossover speed

```
val(:,:,31) =
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
  -0.6027 + 0.0000i   0.6027 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.2185i   0.0000 - 0.2185i
val(:,:,32) =
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
  -0.5766 + 0.0000i   0.5766 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.1295 + 0.0000i  -0.1295 + 0.0000i
```

real valued P vectors above the P crossover speed

PROBLEM 6: LAMINA CHRISTOFFEL EQUATION FOR UNIDIRECTIONAL COMPOSITE LAMINA

Given a unidirectional composite lamina of thickness *h*, solve the lamina Christoffel equation for guided wave propagation along the fiber direction x_1. Do the following:

(a) For a given wavespeed v, set up the cubic equation α^2, where α^2 are the partial-wave slowness coefficients
(b) Solve the cubic equation, find the three α^2 roots $\alpha_I^2, \alpha_{II}^2, \alpha_{III}^2$, and the corresponding six eigenvalues $\alpha^{(1)} = +\alpha_I$, $\alpha^{(2)} = -\alpha_I$, $\alpha^{(3)} = +\alpha_{II}$, $\alpha^{(4)} = -\alpha_{II}$, $\alpha^{(5)} = +\alpha_{III}$, $\alpha^{(6)} = -\alpha_{III}$
(c) Substitute each eigenvalue $\alpha^{(i)}, i = 1,...,6$ into the Christoffel matrix and find the corresponding eigenvectors $\mathbf{U}^{(i)}$, $i = 1,...,6$
(d) Repeat for other values of v
(e) Discuss your findings

Numerical example: T300/914 CFRP, Table 3

<u>Solution</u>

(a) Setup the cubic equation in partial-wave slowness coefficients α^2: Recall Eq. (4.37), i.e.,

$$A_6\alpha^6 + A_4\alpha^4 + A_2\alpha^2 + A_0 = 0 \tag{4.37}$$

with the coefficients A_6, A_4, A_2, A_0 given by Eqs. (4.38)—(4.41), i.e.,

$$A_6 = C_{33}C_{44}C_{55} - C_{33}C_{45}^2 \tag{4.38}$$

$$\begin{aligned} A_4 = {} & (C_{55}C_{44} - C_{45}^2)(C_{55} - \rho v^2) + C_{55}C_{33}(C_{66} - \rho v^2) + C_{44}C_{33}(C_{11} - \rho v^2) \\ & -2C_{16}C_{45}C_{33} + 2\left(C_{45} + C_{36}\right)\left(C_{13} + C_{55}\right)C_{45} - \left(C_{13} + C_{55}\right)^2 C_{44} - \left(C_{45} + C_{36}\right)^2 C_{55} \end{aligned} \tag{4.39}$$

$$\begin{aligned} A_2 = {} & C_{33}(C_{11} - \rho v^2)(C_{66} - \rho v^2) + C_{44}(C_{11} - \rho v^2)(C_{55} - \rho v^2) + C_{55}(C_{66} - \rho v^2)(C_{55} - \rho v^2) \\ & -(C_{11} - \rho v^2)\left(C_{45} + C_{36}\right)^2 - (C_{66} - \rho v^2)\left(C_{13} + C_{55}\right)^2 - 2(C_{55} - \rho v^2)C_{16}C_{45} \\ & +2C_{16}\left(C_{45} + C_{36}\right)\left(C_{13} + C_{55}\right) - C_{16}^2 C_{33} \end{aligned} \tag{4.40}$$

$$A_0 = \left[(C_{11} - \rho v^2)(C_{66} - \rho v^2) - C_{16}^2\right](C_{55} - \rho v^2) \tag{4.41}$$

(b) Solve Eq. (4.37) using a polynomial roots algorithm available in MATLAB to get $\alpha_I^2, \alpha_{II}^2, \alpha_{III}^2$; then, separate the six individual eigenvalues $\alpha^{(1)} = +\alpha_I$, $\alpha^{(2)} = -\alpha_I$, $\alpha^{(3)} = +\alpha_{II}$, $\alpha^{(4)} = -\alpha_{II}$, $\alpha^{(5)} = +\alpha_{III}$, $\alpha^{(6)} = -\alpha_{III}$

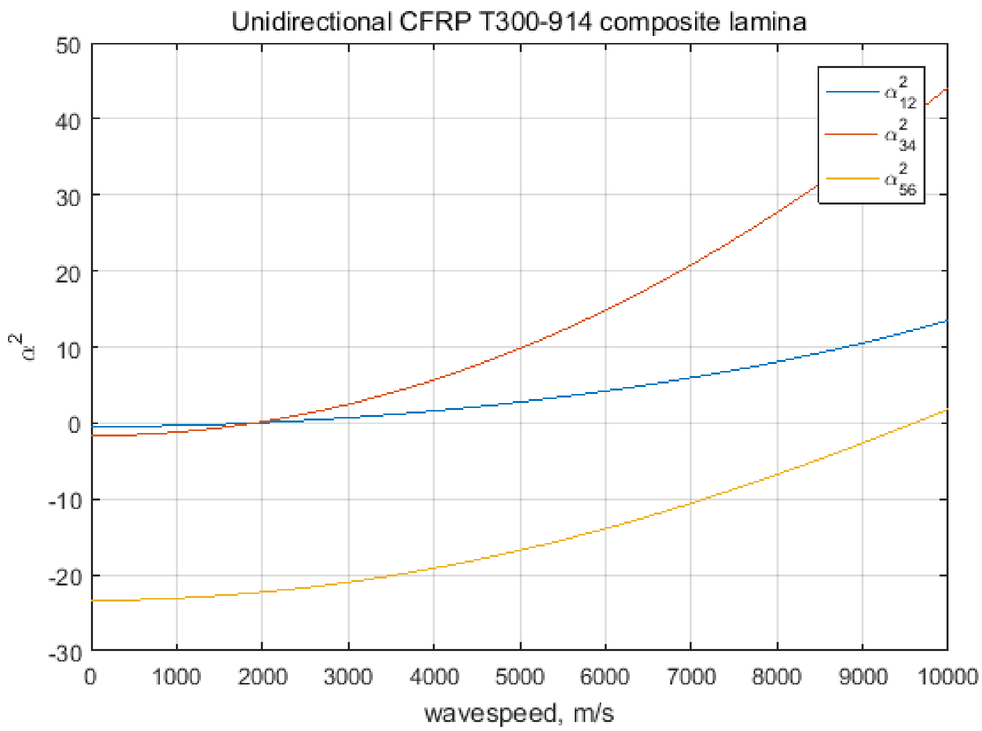
(c) Substitute each eigenvalue $\alpha^{(i)}, i = 1,...,6$ into the Christoffel matrix of Eq. (4.35), i.e.,

$$\begin{bmatrix} \left(C_{11}-\rho v^2\right)+C_{55}\alpha^2 & C_{16}+C_{45}\alpha^2 & \left(C_{13}+C_{55}\right)\alpha \\ C_{16}+C_{45}\alpha^2 & \left(C_{66}-\rho v^2\right)+C_{44}\alpha^2 & \left(C_{36}+C_{45}\right)\alpha \\ \left(C_{13}+C_{55}\right)\alpha & \left(C_{36}+C_{45}\right)\alpha & \left(C_{55}-\rho v^2\right)+C_{33}\alpha^2 \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.35}$$

Find the corresponding eigenvector $\mathbf{U}^{(i)},\ i=1,\ldots,6$ using a standard algebraic eigen problem algorithm (e.g., eig) or a singular value decomposition (SVD) algorithm and choosing the eigenvector corresponding to the eigenvalue which has a value zero.

(d) The above analysis is repeated for all wavespeed v values

The plot below shows the variation of α^2 with wavespeeds.



(e) Discussion:

<u>The α^2 Curves</u>

(e1) Examination of the α^2 curves reveals three monotonically increasing curves. Two of these curves, α_{12}^2 and α_{34}^2, seem to be related and intersect each other at the point where they cross over from the negative lower domain into the positive upper domain. We assume that these two waves are the quasi S waves. The third curve, α_{56}^2, is quite separate; we assume this to be the quasi P wave.

(e2) The quasi P curve stars at around $\alpha_{56}^2 \approx -23$ at almost zero wavespeed[2] ($v \approx 0$) and increases monotonically crossing the horizontal axis at $v_P \approx 10{,}000$ m/s.

(e3) The quasi S curves start much closer to zero and also increase monotonically, one faster than the other. They cross the horizontal axis simultaneously at a wavespeed around $v_S \approx 2{,}000$ m/s.

(e4) At low wavespeeds below the crossover wavespeed around 2,000 m/s, the quasi SH α_{34}^2 curve has middle position among the curves shown on the plot. As the wavespeed passes the crossover point, the quasi SH curve α_{34}^2 switches from the middle to the top position on the plot.

<u>The Polarization Vectors</u>

(e4) Examination of the polarization vectors are such that a quasi SH wave can be identified. The quasi-SH wave is associated with α_{34}^2. The SH wave is polarized in the x_2 direction, i.e., its polarization vector only contains the u_2 component. The quasi SH wave is orthogonal onto the other two waves, the quasi SV wave α_{12}^2 and the quasi P wave α_{56}^2 which contain only the u_1, u_3 displacements. This phenomenon persist for all wavespeeds.

<u>quasi-SH polarization</u>

```
val(:,:,1) =
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0574i   0.0000 + 0.0574i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.1887i   0.0000 - 0.1887i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
val(:,:,2) =
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0571i   0.0000 + 0.0571i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.1887i   0.0000 - 0.1887i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
val(:,:,3) =
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0562i   0.0000 + 0.0562i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.1888i   0.0000 - 0.1888i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
```

[2] exact $v = 0$ is not computable correctly; here, we took $v_{\min} = 10$ m/s

(e5) At all wavespeeds, the quasi SV wave α_{12}^2 and the quasi P wave α_{56}^2, have polarization vectors aligned with the vertical plane $x_1 O x_3$ because they contain only the u_1, u_3 displacements. At low wavespeeds, these vectors are complex.

complex quasi SV polarization vectors

complex quasi P polarization vectors

```
val(:,:,1) =
   0.0000 - 0.0574i   0.0000 + 0.0574i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.1887i   0.0000 - 0.1887i
val(:,:,2) =
   0.0000 - 0.0571i   0.0000 + 0.0571i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.1887i   0.0000 - 0.1887i
val(:,:,3) =
   0.0000 - 0.0562i   0.0000 + 0.0562i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.1888i   0.0000 - 0.1888i
```

(e6) As the wavespeed passes the S crossover point $v_S \approx 2{,}000$ m/s, the SV polarization vectors corresponding to α_1, α_2 switch from complex to real values.

complex quasi SV vectors below the S crossover speed

```
val(:,:,10) =
   0.0000 - 0.0197i   0.0000 + 0.0197i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.1892i   0.0000 - 0.1892i
val(:,:,11) =
  -0.0181 + 0.0000i   0.0181 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.1893i   0.0000 - 0.1893i
```

real valued quasi SV vectors above the S crossover speed

(e7) As the wavespeed passes the P crossover point $v_P \approx 10{,}000$ m/s, the P polarization vectors corresponding to α_5, α_6 switch from complex to real values.

complex quasi P vectors below the P crossover speed

```
val(:,:,49) =
  -0.5963 + 0.0000i   0.5963 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0058i   0.0000 - 0.0058i
val(:,:,50) =
  -0.6340 + 0.0000i   0.6340 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0848 + 0.0000i  -0.0848 + 0.0000i
```

real valued quasi P vectors above the P crossover speed

PROBLEM 7: LAMINA CHRISTOFFEL EQUATION FOR OFF-AXIS COMPOSITE LAMINA

Given a unidirectional composite lamina of thickness *h*, solve the lamina Christoffel equation for guided wave propagation along an off-axis direction. Do the following:

(a) For a given wavespeed v, set up the cubic equation α^2, where α^2 are the partial-wave slowness coefficients

(b) Solve the cubic equation, find the three α^2 roots $\alpha_I^2, \alpha_{II}^2, \alpha_{III}^2$, and the corresponding six eigenvalues $\alpha^{(1)} = +\alpha_I$, $\alpha^{(2)} = -\alpha_I$, $\alpha^{(3)} = +\alpha_{II}$, $\alpha^{(4)} = -\alpha_{II}$, $\alpha^{(5)} = +\alpha_{III}$, $\alpha^{(6)} = -\alpha_{III}$

(c) Substitute each eigenvalue $\alpha^{(i)}, i = 1,...,6$ into the Christoffel matrix and find the corresponding eigenvectors $\mathbf{U}^{(i)}, \; i = 1,...,6$

(d) Repeat for other values of v

(e) Repeat (a) through (e) for other values of θ

(f) Discuss your findings

Numerical example: T300/914 CFRP, Table 3;
Off-axis direction angle $\theta = 0°, 5°, ..., 85°, 90°$, i.e., in $5°$ increments.

Solution

(a) Calculate the rotated stiffness matrix $\mathbf{C}$ using Eqs. (2.179) and (2.193) of Chapter 2.

Setup the cubic equation in partial-wave slowness coefficients α^2: Recall Eq. (4.37), i.e.,

$$A_6\alpha^6 + A_4\alpha^4 + A_2\alpha^2 + A_0 = 0 \tag{4.37}$$

with the coefficients A_6, A_4, A_2, A_0 given by Eqs. (4.38)—(4.41), i.e.,

$$A_6 = C_{33}C_{44}C_{55} - C_{33}C_{45}^2 \tag{4.38}$$

$$\begin{aligned} A_4 &= (C_{55}C_{44} - C_{45}^2)(C_{55} - \rho v^2) + C_{55}C_{33}(C_{66} - \rho v^2) + C_{44}C_{33}(C_{11} - \rho v^2) \\ &\quad -2C_{16}C_{45}C_{33} + 2\left(C_{45} + C_{36}\right)\left(C_{13} + C_{55}\right)C_{45} - \left(C_{13} + C_{55}\right)^2 C_{44} - \left(C_{45} + C_{36}\right)^2 C_{55} \end{aligned} \tag{4.39}$$

$$\begin{aligned} A_2 &= C_{33}(C_{11} - \rho v^2)(C_{66} - \rho v^2) + C_{44}(C_{11} - \rho v^2)(C_{55} - \rho v^2) + C_{55}(C_{66} - \rho v^2)(C_{55} - \rho v^2) \\ &\quad -(C_{11} - \rho v^2)\left(C_{45} + C_{36}\right)^2 - (C_{66} - \rho v^2)\left(C_{13} + C_{55}\right)^2 - 2(C_{55} - \rho v^2)C_{16}C_{45} \\ &\quad +2C_{16}\left(C_{45} + C_{36}\right)\left(C_{13} + C_{55}\right) - C_{16}^2 C_{33} \end{aligned} \tag{4.40}$$

$$A_0 = \left[(C_{11} - \rho v^2)(C_{66} - \rho v^2) - C_{16}^2\right](C_{55} - \rho v^2) \tag{4.41}$$

(b) Solve Eq. (4.37) using a polynomial roots algorithm available in MATLAB to get $\alpha_I^2, \alpha_{II}^2, \alpha_{III}^2$; then, separate the six individual eigenvalues $\alpha^{(1)} = +\alpha_I$, $\alpha^{(2)} = -\alpha_I$, $\alpha^{(3)} = +\alpha_{II}$, $\alpha^{(4)} = -\alpha_{II}$, $\alpha^{(5)} = +\alpha_{III}$, $\alpha^{(6)} = -\alpha_{III}$

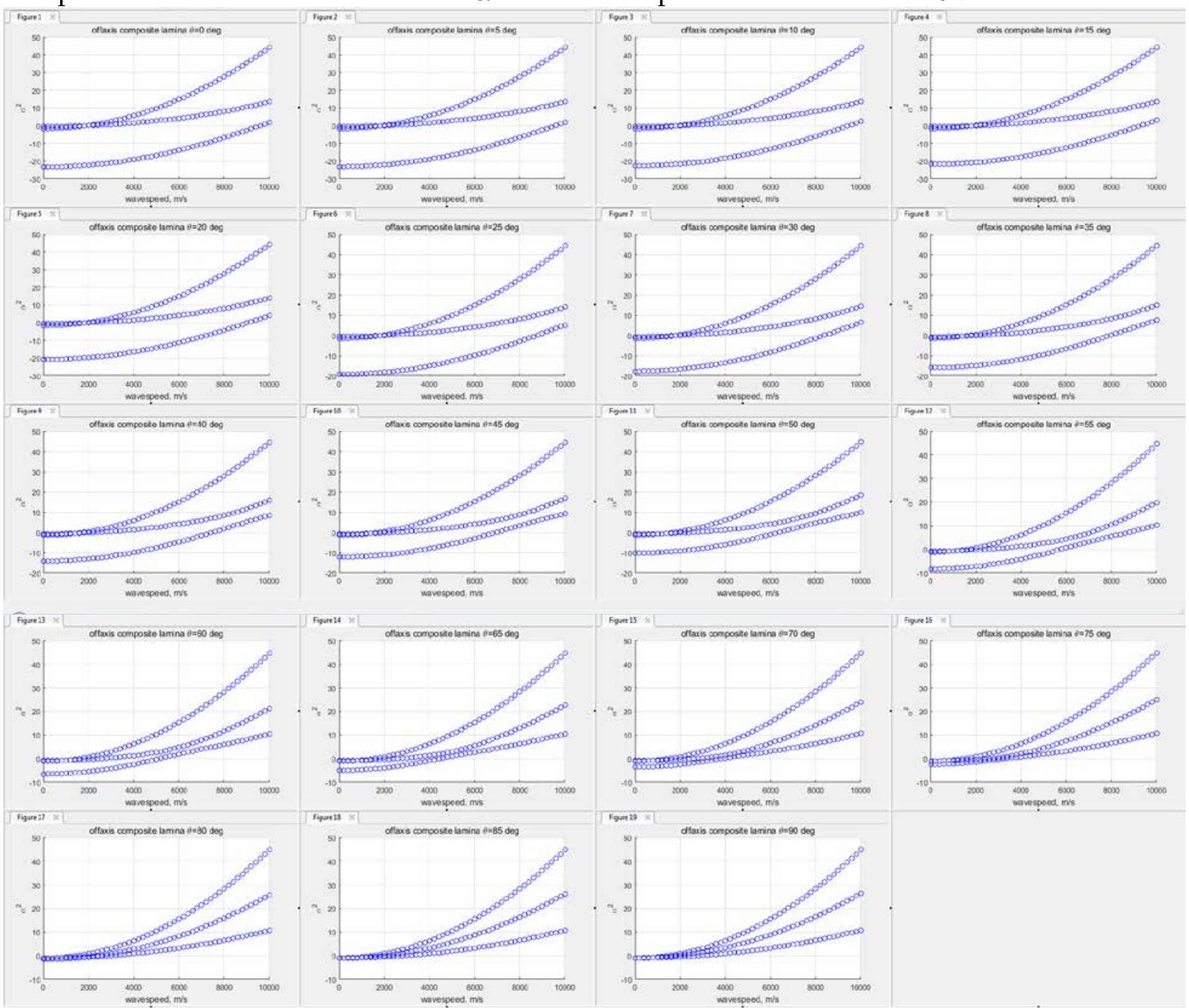
(c) Substitute each eigenvalue $\alpha^{(i)}, i=1,\ldots,6$ into the Christoffel matrix of Eq. (4.35), i.e.,

$$\begin{bmatrix} \left(C_{11}-\rho v^2\right)+C_{55}\alpha^2 & C_{16}+C_{45}\alpha^2 & \left(C_{13}+C_{55}\right)\alpha \\ C_{16}+C_{45}\alpha^2 & \left(C_{66}-\rho v^2\right)+C_{44}\alpha^2 & \left(C_{36}+C_{45}\right)\alpha \\ \left(C_{13}+C_{55}\right)\alpha & \left(C_{36}+C_{45}\right)\alpha & \left(C_{55}-\rho v^2\right)+C_{33}\alpha^2 \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.35}$$

Find the corresponding eigenvector $\mathbf{U}^{(i)}, i=1,\ldots,6$ using a standard algebraic eigen problem algorithm (e.g., eig) or a singular value decomposition (SVD) algorithm and choosing the eigenvector corresponding to the eigenvalue which has a value zero.

(d), (e) The above analysis is repeated for all values of wavespeed v and off-axis angle θ

The plot below shows the variation of α^2 with wavespeeds for 19 values of θ.



(f) Discussion:

The α^2 Curves

(f1) At low values of the off-axis angle θ, the behavior of the α^2 curves resemble the behavior for an unidirectional lamina at $\theta = 0°$, i.e., one can distinguish a quasi P wave and two quasi S waves. The P wave stars at around $\alpha_I^2 \approx -23$ at almost zero wavespeed[3] ($v \approx 0$) and increases monotonically crossing the horizontal axis at high wavespeed values ($v \approx 10{,}000$ m/s). The $v \approx 0$ value of the quasi P wave decreases with increasing off-axis angle θ. The wavespeed at which the quasi P wave crosses over the horizontal axis also decreases with increasing off-axis angle θ.

The S waves start much closer to zero and also increase monotonically, one faster than the other. They cross the horizontal axis simultaneously at a wavespeed around 2,000 m/s; this crossover wavespeed decreases as the off-axis angle increases.

(f2) as the off-axis angle θ increases, one observes a tendency for quasi P wave curves and one of the quasi S wave curve to come together at higher wavespeed values

(f3) at high values of the off-axis angle θ, the P wave curve and the S wave curves come together at lower wavespeeds

(f4) at $\theta = 90°$, all three α^2 curves stem for the same point just below the horizontal axis at $v = 0$ and then fan out.

The Polarization Vectors

(f5) at $\theta = 0°$, the polarization vectors are such that a quasi SH wave can be identified. This phenomenon persist for all wavespeeds.

Vectors{1, 1}

quasi-SH polarization

```
val(:,:,1) =
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0574i   0.0000 + 0.0574i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.1887i   0.0000 - 0.1887i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
val(:,:,2) =
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0571i   0.0000 + 0.0571i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.1887i   0.0000 - 0.1887i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
val(:,:,3) =
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0562i   0.0000 + 0.0562i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.1888i   0.0000 - 0.1888i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
```

(f6) at low values of the off-axis angle θ, the polarization vectors DO NOT resemble the behavior for an unidirectional lamina with $\theta = 0°$. The main difference is that the polarization vectors become coupled as soon as the off-axis angle has a non-zero value. Hence, it is not possible to identify a quasi SH wave as it was possible for the unidirectional lamina ($\theta = 0°$). This behavior persist throughout all the off-axis angle θ with the exception of $\theta = 90°$.

---

[3] exact $v = 0$ is not computable correctly

(f7) at $\theta = 90°$, when all three α^2 curves stem for the same point just below the horizontal axis at $\upsilon \approx 0$, the polarization vectors have again become decouples and one can identify a quasi SH wave for $\upsilon > 0$. (Note that at $\upsilon \approx 0$, the polarization vectors are all identical because the α^2 curves stem for the same point at $\upsilon \approx 0$.)

Vectors{19, 1}

```
val(:,:,1) =
   0.0000 - 1.0000i   0.0000 + 1.0000i   0.0000 - 1.0000i   0.0000 + 1.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 1.0000i   0.0000 - 1.0000i
val(:,:,2) =
   1.0000 + 0.0000i   1.0000 + 0.0000i  -0.0000 + 0.0000i  -0.0000 + 0.0000i   0.0000 - 0.9908i   0.0000 + 0.9908i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.9977i   0.0000 - 0.9977i   0.0000 - 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
val(:,:,3) =
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.9626i   0.0000 + 0.9626i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.9906i   0.0000 - 0.9906i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
```

quasi-SH polarization

PROBLEM 8: LAMINA CHRISTOFFEL EQUATION IN ORTHOTROPIC COMPOSITE LAMINA

Given a fully orthotropic composite lamina of given thickness *h*, solve the lamina Christoffel equation for guided wave propagation along the fiber direction. Do the following:

(a) For a given wavespeed v, set up the cubic equation in partial wave slowness coefficients α^2

(b) Solve the cubic equation, find the three α^2 roots $\alpha_I^2, \alpha_{II}^2, \alpha_{III}^2$, and the corresponding six eigenvalues $\alpha^{(1)} = +\alpha_I$, $\alpha^{(2)} = -\alpha_I$, $\alpha^{(3)} = +\alpha_{II}$, $\alpha^{(4)} = -\alpha_{II}$, $\alpha^{(5)} = +\alpha_{III}$, $\alpha^{(6)} = -\alpha_{III}$

(c) Substitute each eigenvalue $\alpha^{(i)}, i = 1,...,6$ into the Christoffel matrix and find the corresponding eigenvectors $\mathbf{U}^{(i)},\ i = 1,...,6$

(d) Repeat for other values of v

(e) Discuss your findings

Numerical example: Fully orthotropic CFRP composite, Table 4

Solution

(a) Setup the cubic equation in partial-wave slowness coefficients α^2: Recall Eq. (4.37), i.e.,

$$A_6\alpha^6 + A_4\alpha^4 + A_2\alpha^2 + A_0 = 0 \tag{4.37}$$

with the coefficients A_6, A_4, A_2, A_0 given by Eqs. (4.38)—(4.41), i.e.,

$$A_6 = C_{33}C_{44}C_{55} - C_{33}C_{45}^2 \tag{4.38}$$

$$\begin{aligned} A_4 = {} & (C_{55}C_{44} - C_{45}^2)(C_{55} - \rho v^2) + C_{55}C_{33}(C_{66} - \rho v^2) + C_{44}C_{33}(C_{11} - \rho v^2) \\ & -2C_{16}C_{45}C_{33} + 2\left(C_{45} + C_{36}\right)\left(C_{13} + C_{55}\right)C_{45} - \left(C_{13} + C_{55}\right)^2 C_{44} - \left(C_{45} + C_{36}\right)^2 C_{55} \end{aligned} \tag{4.39}$$

$$\begin{aligned} A_2 = {} & C_{33}(C_{11} - \rho v^2)(C_{66} - \rho v^2) + C_{44}(C_{11} - \rho v^2)(C_{55} - \rho v^2) + C_{55}(C_{66} - \rho v^2)(C_{55} - \rho v^2) \\ & -(C_{11} - \rho v^2)\left(C_{45} + C_{36}\right)^2 - (C_{66} - \rho v^2)\left(C_{13} + C_{55}\right)^2 - 2(C_{55} - \rho v^2)C_{16}C_{45} \\ & +2C_{16}\left(C_{45} + C_{36}\right)\left(C_{13} + C_{55}\right) - C_{16}^2 C_{33} \end{aligned} \tag{4.40}$$

$$A_0 = \left[(C_{11} - \rho v^2)(C_{66} - \rho v^2) - C_{16}^2\right](C_{55} - \rho v^2) \tag{4.41}$$

(b) Solve Eq. (4.37) using a polynomial roots algorithm available in MATLAB to get $\alpha_I^2, \alpha_{II}^2, \alpha_{III}^2$; then, separate the six individual eigenvalues $\alpha^{(1)} = +\alpha_I$, $\alpha^{(2)} = -\alpha_I$, $\alpha^{(3)} = +\alpha_{II}$, $\alpha^{(4)} = -\alpha_{II}$, $\alpha^{(5)} = +\alpha_{III}$, $\alpha^{(6)} = -\alpha_{III}$

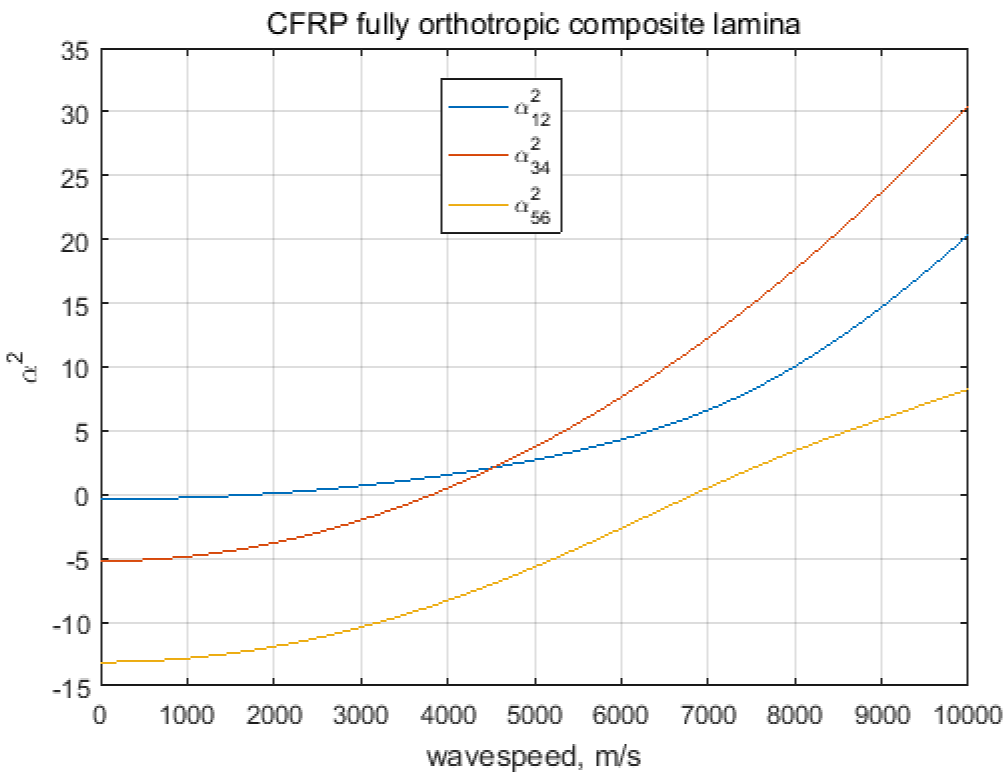
(c) Substitute each eigenvalue $\alpha^{(i)}, i = 1,...,6$ into the Christoffel matrix of Eq. (4.35), i.e.,

$$\begin{bmatrix} \left(C_{11}-\rho v^2\right)+C_{55}\alpha^2 & C_{16}+C_{45}\alpha^2 & \left(C_{13}+C_{55}\right)\alpha \\ C_{16}+C_{45}\alpha^2 & \left(C_{66}-\rho v^2\right)+C_{44}\alpha^2 & \left(C_{36}+C_{45}\right)\alpha \\ \left(C_{13}+C_{55}\right)\alpha & \left(C_{36}+C_{45}\right)\alpha & \left(C_{55}-\rho v^2\right)+C_{33}\alpha^2 \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.35}$$

Find the corresponding eigenvector $\mathbf{U}^{(i)}$, $i=1,...,6$ using a standard algebraic eigen problem algorithm (e.g., eig) or a singular value decomposition (SVD) algorithm and choosing the eigenvector corresponding to the eigenvalue which has a value zero.

(d) The above analysis is repeated for all wavespeed v values

The plot below shows the variation of α^2 with wavespeeds.



(e) Discussion:

The α^2 Curves

(e1) Examination of the α^2 curves reveals three monotonically increasing curves. Two of these curves, α_{12}^2 and α_{34}^2, seem to be related. We assume that these are the quasi SV and SH curves that were observed in a unidirectional lamina. However, the intersection of these quasi S curves does not take place when they cross the horizontal axis, but later which is different from the behavior observed in the unidirectional lamina.

(e2) The quasi S wave curves cross the horizontal axis at two different points, one at $v_{SV} \approx 1{,}700$ m/s, the other at $v_{SH} \approx 3{,}500$ m/s (the assignation of 'SH' and 'SV' labels is done using the polarization vectors, see (e5) below).

(e3) At low wavespeeds, the quasi SH α_{34}^2 curve has middle position among the curves shown on the plot. As the wavespeed passes the intersection point, $v_{SH=SV} \approx 5{,}000$ m/s, the quasi SH curve α_{34}^2 switches from the middle to the top position on the plot.

(e4) The quasi P wave curve starts well below the quasi S curves and also increases monotonically. It crosses the horizontal axis around $v_P \approx 7{,}000$ m/s.

The Polarization Vectors

(e5) Examination of the polarization vectors are such that a quasi SH wave can be identified. The quasi-SH wave is associated with the α_{34}^2. The SH wave is polarized in the x_2 direction, i.e., its polarization vector only contains the u_2 component. The quasi SH wave is orthogonal onto the other two waves, the quasi SV wave α_{12}^2 and the quasi P wave α_{56}^2 which contain only the u_1, u_3 displacements, i.e., have polarization vectors aligned with the vertical plane $x_1 \mathrm{O} x_3$. This phenomenon is observed at all wavespeeds.

quasi-SH polarization

```
val(:,:,1) =
   0.0000 - 0.0961i   0.0000 + 0.0961i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.2097i   0.0000 - 0.2097i
val(:,:,2) =
   0.0000 - 0.0956i   0.0000 + 0.0956i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.2098i   0.0000 - 0.2098i
val(:,:,3) =
   0.0000 - 0.0939i   0.0000 + 0.0939i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.2099i   0.0000 - 0.2099i
```

(e5) At low wavespeeds, quasi SV and quasi P vectors are complex.

complex quasi SV polarization vectors

complex quasi P polarization vectors

```
val(:,:,1) =
   0.0000 - 0.0961i   0.0000 + 0.0961i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.2097i   0.0000 - 0.2097i
val(:,:,2) =
   0.0000 - 0.0956i   0.0000 + 0.0956i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.2098i   0.0000 - 0.2098i
val(:,:,3) =
   0.0000 - 0.0939i   0.0000 + 0.0939i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.2099i   0.0000 - 0.2099i
```

(e6) As the wavespeed passes the SV crossover point $v_{SV} \approx 1{,}700$ m/s, the SV polarization vectors corresponding to α_1, α_2 switch from complex to real values.

complex quasi SV vectors below the SV crossover speed

```
val(:,:,9) =
   0.0000 - 0.0427i   0.0000 + 0.0427i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.2120i   0.0000 - 0.2120i
val(:,:,10) =
  -0.0186 + 0.0000i   0.0186 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.2126i   0.0000 - 0.2126i
```

real valued quasi SV vectors above the SV crossover speed

(e7) As the wavespeed passes the P crossover point $v_P \approx 7{,}000$ m/s, the P polarization vectors corresponding to α_5, α_6 switch from complex to real values.

complex quasi P vectors below the P crossover speed

```
val(:,:,35) =
  -0.9199 + 0.0000i   0.9199 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0517i   0.0000 - 0.0517i
val(:,:,36) =
   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
  -0.9844 + 0.0000i   0.9844 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.1296 + 0.0000i  -0.1296 + 0.0000i
```

real valued quasi P vectors above the P crossover speed